Inverse Problems in Imaging - Coursework 3

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1 Introduction

This report is for the third coursework in Inverse Problems in Imaging for the academic year of 2019 to 2020. It looks at the paper Stable Signal Recovery from Incomplete and Inaccurate Measurements [1]. The first section will provide an analysis of the aforementioned paper and discuss its influence. The latter section will discuss the results found from any numerical experiments we undertake from the information provided in the paper. The experiments carried out in this report uses MATLAB exclusively and the source code can be found in the appendix.

2 Reconstruction with Sparsity Constraints

2.1 Critical Analysis

Much like many inverse problems, this paper poses the problem as a system of linear equations (SLE). Fundamentally, is it possible to recover x_0 from the equation $y = Ax_0 + e$?

$$
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} A_{1,1} A_{1,2} \ldots A_{1,n} \\ A_{2,1} A_{2,2} \ldots A_{2,n} \\ \vdots \\ A_{n,1} A_{n,2} \ldots A_{n,n} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \tag{1}
$$

Equation (1) sets up the problem. We would like to fully recover the signal x_0, where there are are more unknowns than there are observations. We can refer to this as an underdetermined system of equations. Furthermore, this equation also considers the very likely possibility of error in the form of noise in our measurements. Regardless, this research argues that we can still recover almost all of x_0 through l_1 minimisation.

"How influential has this paper been?"

As of writing, Google Scholar states that this research has been cited by 7002[1] later papers. It undoubtedly seems that this is quite a significant contribution to the area of signal processing. The applications of this paper spread far and wide, from Robust Face Recognition via Sparse Representation [3] to Sparse MRI: The application of compressed sensing for rapid MR imaging [2].

In the latter paper [2], we can begin to see the impact of stable recovery in MRI (Magnetic Resonance Imaging). MRI inherently works in the spatial frequency, as it measures Fourier coefficients unlike raster information such as pixels in spatial imaging. This is achieved by tracking the protons in the hydrogen atoms of human tissue: as they shift and realign due to the incident radio waves (the y term in Equation 1). Magnetic sensors can pick up this realignment from the millions of signals emitting from these protons and begin to reconstruct the tissue (the result, otherwise known as the x_0 term in Equation 1). Many of these resulting signals can become distorted, or just missing - we refer to this as sparse and with noise. This paper [2] was particularly novel, it showed that the sparsity of MR images could be used to improve image resolution or reduce scan time. This research went on to be cited 5450 times[2].

Moreover, prior to [1], Nyquist and Shannon proved that stable recovery can occur if the average sample rate is of twice the bandwidth. However, Candès et al. provides an alternative, whereby stable recovery can occur "for almost any set of n coefficients provided that the number of non-zeros is of the order of $\frac{n}{log(m)^6}$" [1] (where n and m represents the dimensions of the image A). We will talk more about this later.

[1] Link to Google Scholar results for Candès et al.
[2] Link to Google Scholar results for Lustig et al.

"How and under what circumstances can a signal f be exactly reconstructed from a discrete set of samples?"

If our matrix A obeys a uniform uncertainty principle then we are able to solve the following minimisation problem and completely recover x_0:

$$\min ||x||_{l1} \text{ subject to } Ax = y$$

This means that A obeys a restricted isometry hypothesis. This is however for the unlikely situation of collecting data without noise - perfect measurement. An imperfect measurement, which is much more likely to occur, requires us to solve the following:

$$\min ||x||_{l1} \text{ subject to } ||Ax - y||_{l2} \leq \epsilon$$

The solution to this recovers an unknown sparse object with an error proportional to the noise ϵ. Again, this still requires restricted isometry constants. The paper [1] refers to many example matrices that obeys this hypothesis, such as:

- Random matrices with independent and identically distributed entries.
- Fourier ensemble. Previously we mentioned how MRI scans pickup Fourier coefficients; this is why this research is particularly useful as these coefficients form an orthogonal set. Thus, any orthogonal ensemble can be used.
- General orthogonal measurement ensemble.

2.2 Numerical Experiments

In our experiment, consider an signal f of length 1024 x 1 and a result y of length 300 x 1 observations. Therefore, this is experiment is measured with a Gaussian measurement ensemble (image) A of size 300 x 1024.

$$y = Af \tag{2}$$

We pose this as a minimisation problem to solve:

$$\min ||f||_1, \text{ subject to } ||Af - y||_2 <= \epsilon \tag{3}$$

Using the L1-Magic MATLAB package, we were able to recreate the experiment found in the paper. The reconstruction (shown below) was very good. We were almost perfectly able to reconstruct the original signal f using the minimisation equation.

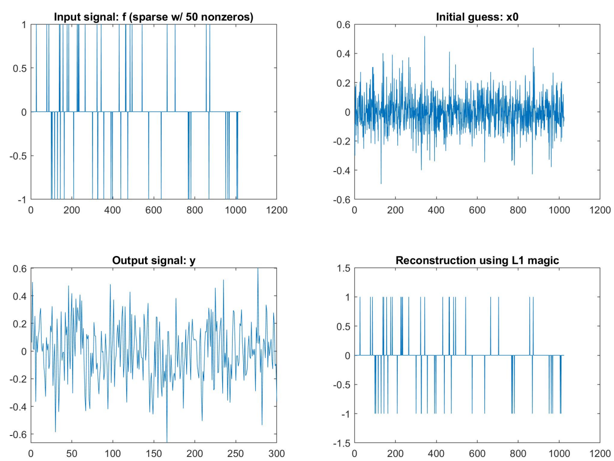

Figure 1: Upper-left: the sparse signal f with 50 non-zero values. Upper-right: the signal multiplied to our measurement ensemble. Bottom-left: the output signal y. Bottom-right: the reconstruction recovered from noisy measurements with 11 iterations of L1-Magic.

In comparison, let us compare this solution to a more common Least-Squares approach. This is solved as the following:

$$(A^*_{T0}A_{T0})^{-1}A^*_{T0}y$$

The output of this below shows that recovery of the original signal y was almost non-existent.

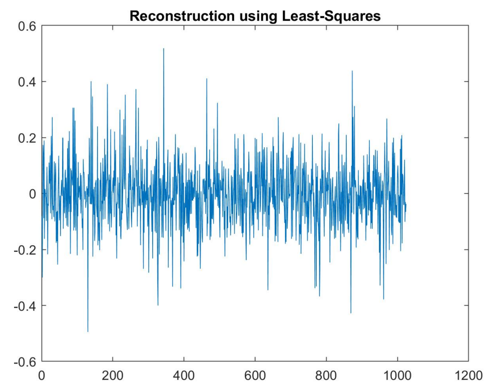

Figure 2: A Least-Squares reconstruction recovered from noisy measurements.

To conclude, Figure 2 shows that L1 minimisation works incredibly well in circumstances where the matrix obeys a uniform uncertainty principle. We are able to recover the original signal almost exactly. The applications of such a tool can be used in most inverse problems where the original signal needs to be recovered. This is incredibly relevant today as this is a computationally efficient approach, only requiring 11 iterations for this particular toy problem.

3 Appendix

3.1 MATLAB Implementation

```
%% Inverse Problems in Imaging
% Coursework 3 - Numerical Experiment
% Jaspreet Singh Dhanjan

% Signal length
N = 1024;

% Observations
K = 300;

% Gaussian matrix
A = rand(K, N);
U = orth(A');
V = U';

% Number of non-zero coeffecients
c = 50;

% Signal
f = zeros(N, 1);
perm = randperm(N);
f(perm(1:c)) = sign(randn(c, 1));

% Apply to experiment
y = V * f;

% Initial guess
x0 = V' * y;

% L1 magic
xp = l1eq_pd(x0, V, [], y, 1e-3);

figure;
subplot(2, 2, 1); plot(f); title('Input signal: f (sparse w/ 50 nonzeros)');
subplot(2, 2, 2); plot(x0); title('Initial guess: x0');
subplot(2, 2, 3); plot(y); title('Output signal: y');
subplot(2, 2, 4); plot(xp); title('Reconstruction using L1 magic');
```

3.2 L1 Console Output

Iteration = 1, tau = 1.604e+02, Primal = 1.411e+02, PDGap = 1.277e+02, Dual res = 8.836e+00, Primal res = 1.752e-14 H11p condition number = 3.604e-03

Iteration = 2, tau = 2.768e+02, Primal = 1.119e+02, PDGap = 7.398e+01, Dual res = 4.473e+00, Primal res = 8.962e-15 H11p condition number = 5.362e-03

Iteration = 3, tau = 4.199e+02, Primal = 9.421e+01, PDGap = 4.878e+01, Dual res = 2.733e+00, Primal res = 1.402e-14 H11p condition number = 9.848e-05

Iteration = 4, tau = 4.942e+02, Primal = 8.833e+01, PDGap = 4.144e+01, Dual res = 2.272e+00, Primal res = 1.357e-14 H11p condition number = 6.534e-05

Iteration = 5, tau = 6.735e+02, Primal = 7.882e+01, PDGap = 3.041e+01, Dual res = 1.591e+00, Primal res = 1.109e-14 H11p condition number = 5.806e-05

Iteration = 6, tau = 1.381e+03, Primal = 6.447e+01, PDGap = 1.483e+01, Dual res = 6.687e-01, Primal res = 2.549e-14 H11p condition number = 1.748e-05

Iteration = 7, tau = 1.122e+04, Primal = 5.180e+01, PDGap = 1.825e+00, Dual res = 7.515e-03, Primal res = 2.081e-13 H11p condition number = 1.727e-06

Iteration = 8, tau = 1.028e+05, Primal = 5.019e+01, PDGap = 1.992e-01, Dual res = 7.515e-05, Primal res = 1.188e-12 H11p condition number = 2.990e-08

Iteration = 9, tau = 9.432e+05, Primal = 5.002e+01, PDGap = 2.171e-02, Dual res = 7.515e-07, Primal res = 3.386e-12 H11p condition number = 2.020e-09

Iteration = 10, tau = 8.653e+06, Primal = 5.000e+01, PDGap = 2.367e-03, Dual res = 7.515e-09, Primal res = 2.640e-11 H11p condition number = 2.605e-11

Iteration = 11, tau = 7.939e+07, Primal = 5.000e+01, PDGap = 2.580e-04, Dual res = 7.515e-11, Primal res = 2.048e-10 H11p condition number = 3.140e-13

References

[1] Emmanuel J. Candès, Justin K. Romberg, and Terence Tao. Stable signal recovery from incomplete and inaccurate measurements. *Communications on Pure and Applied Mathematics*, 59(8):1207–1223, 2006.

[2] Michael Lustig, David Donoho, and John M. Pauly. Sparse mri: The application of compressed sensing for rapid mr imaging. *Magnetic Resonance in Medicine*, 58(6):1182–1195, 2007.

[3] J. Wright, A. Y. Yang, A. Ganesh, S. S. Sastry, and Y. Ma. Robust face recognition via sparse representation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 31(2):210–227, 2 2009.